A Method for the Analysis of Biaxial Graded-Index Optical Fibers

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Abstract—**The problem of wave propagation in a biaxial graded-index fiber with circular symmetry is considered. The problem is formulated in terms of four first-order differential equations for the tangential components of the electric and magnetic fields. A general solution method for solving systems of differential equations is presented. This solution method is then used to solve the system of equations for a particular example of a biaxial graded-index fiber. Numerical results for the propagation constant in the fiber are also given.**

I. Introduction

The optical fiber has become a much-studied transmission system owing to its property of wave guidance with low loss. In recent years it has been shown that introducing anisotropies into the dielectric medium of the fiber produces several interesting features, such as control of power flow [1], loss characteristics [2], and reduction of peak attenuation near cutoff [3].

Typically the analysis of wave propagation in a cylindrical dielectric waveguide such as an optical fiber is performed using a wave equation formulation. For the simple case of a step-index fiber a detailed analysis, including dispersion relations, cutoff conditions, and mode designations, is presented by Snitzer [4]. Paul and Shevgaonkar [3] present a similar analysis for a uniaxial step-index fiber and also perform a perturbation analysis to determine the modal attenuation constants. These are the only two cases for which exact solutions are known.

For inhomogeneous fibers no exact solutions are known. For the case of an isotropic graded-index fiber several approximate analytic solution methods are available. These approximate solutions all share the common assumption that the fiber is infinite in extent. In addition if the permittivity is assumed to vary slowly over the distance of one wavelength the wave equation formulation simplifies to an associated scalar wave equation. If the permittivity profile is parabolic, the solution to the scalar wave equation can be written in terms of either Laguerre polynomials if cylindrical coordinates are used or Hermite polynomials if rectangular coordinates are used [5]. For arbitrary permittivity profiles the scalar wave equation can be solved using the well-known Wentzel–Kramers–Brillouin (WKB) solution method [6], [7]. For a parabolic permittivity profile all three solution methods give identical results. Under the assumption that the fields are far from cutoff, Kurtz and Streifer [8], [9] have shown that a solution to the full vector problem can be written either in terms of Laguerre polynomials if the permittivity profile is quadratic or asymptotically in terms of Bessel and Airy functions for arbitrary permittivity profiles which decrease slowly and monotonically. A comparison of the vector and scalar solutions for the quadratic permittivity profile implies that the vector modes can be obtained by simply renumbering the scalar modes [10]. Using the renumbered scalar modes as a basis, Hashimoto [11] and Ikuno [12] have developed two slightly different asymptotic methods which can be used to solve the full vector problem for an isotropic graded-index fiber.

These approximate solutions all share the common feature that the propagation constants are determined during the process of finding a solution to the wave equation. This differs from the case of a step-index fiber, where the propagation constants are determined by using the solutions to the wave equation to impose the electromagnetic boundary conditions at the interface between the core and the cladding of the optical fiber. Comparing the propagation constants obtained using one of the approximate solution methods with those obtained using a numerical solution method [13], one finds that these approximate solutions methods produce modes which do not appear in the numerical solution. This suggests that the electromagnetic boundary conditions at the core–cladding interface are important and that an alternative formulation which permits the imposition of the boundary conditions is needed.

An alternative formulation of the problem is to write the four first-order differential equations for the tangential field components as a first-order matrix differential equation. For a step-index fiber with uniaxial core and cladding, Tonning [14] has shown that the matrix formulation can be solved exactly in terms of Bessel functions. For isotropic graded-index fibers with arbitrary permittivity profiles, Yeh and Lingren [13] have indirectly used the matrix formulation in developing a numerical solution method based on the concept of stratification. Using the concept of transition matrices, Tonning [15] has developed a numerical procedure which can be used to solve

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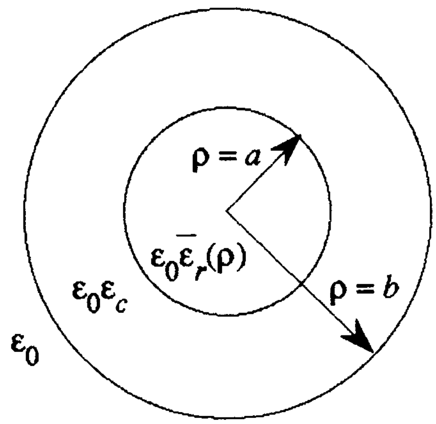

Fig. 1. Geometry of the fiber.

the matrix differential equation for isotropic graded-index fibers.

II. Formulation of the Problem

Consider a circularly symmetric optical fiber with the geometry shown in Fig. 1. In the core, $0 \leqslant \rho \leqslant a$, the permittivity is given by $\epsilon_0 \bar{\epsilon}_r(\rho)$, where ϵ_0 is the permittivity of free space and $\bar{\epsilon}_r(\rho)$ is the relative permittivity tensor of the core and is a function of ρ only. In the cladding, $a \leqslant \rho \leqslant b$, the permittivity is given by $\epsilon_0 \epsilon_c$, where ϵ_c is the relative permittivity of the cladding and is assumed to be constant. In both the core and the cladding the permeability is μ_0, the permeability of free space. For convenience, the external radius of the cladding, b, is assumed to be sufficiently large in comparison with the radius of the core, a, so that it is not necessary to impose boundary conditions at the air–cladding boundary.

Consider the case where the relative permittivity tensor in the core is given by

$$\bar{\epsilon}_r(\rho) = \begin{pmatrix} \epsilon_1(\rho) & 0 & 0 \\ 0 & \epsilon_2(\rho) & 0 \\ 0 & 0 & \epsilon_3(\rho) \end{pmatrix}_{\rho,\phi,z} \tag{1}$$

where $\epsilon_1(\rho)$, $\epsilon_2(\rho)$, and $\epsilon_3(\rho)$ are the relative permittivities in the ρ, ϕ, and z directions respectively. In general the relative permittivities are arbitrary functions of ρ. However, the choice of cylindrical coordinates requires that $\epsilon_1(\rho)$ and $\epsilon_2(\rho)$ be equal at $\rho = 0$.

For time harmonic fields in a source-free region, Maxwell's equations can be written as

$$\nabla \times \boldsymbol{H} = j\omega\epsilon_0 \bar{\epsilon}_r \boldsymbol{E} \tag{2a}$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H} \tag{2b}$$

where ω is the angular frequency. If the z and ϕ dependence of the fields is given by

$$e^{-j\beta z + jm\phi}$$

where β is the longitudinal wavenumber and m is any integer, then for cylindrical coordinates Maxwell's equations can be written in component form as

$$\frac{m}{\rho} H_z + \beta H_\phi = \omega\epsilon_0\epsilon_1 E_\rho \tag{3a}$$

$$-j\beta H_\rho - \frac{dH_z}{d\rho} = j\omega\epsilon_0\epsilon_2 E_\phi \tag{3b}$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi) - \frac{jm}{\rho} H_\rho = j\omega\epsilon_0\epsilon_3 E_z \tag{3c}$$

$$\frac{m}{\rho} E_z + \beta E_\phi = -\omega\mu_0 H_\rho \tag{3d}$$

$$j\beta E_\rho + \frac{dE_z}{d\rho} = j\omega\mu_0 H_\phi \tag{3e}$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho E_\phi) - \frac{jm}{\rho} E_\rho = -j\omega\mu_0 H_z. \tag{3f}$$

The remainder of the problem can now be formulated in two different ways. If the transverse field components E_ρ, E_ϕ, H_ρ, and H_ϕ are eliminated from equations (3), we obtain a pair of coupled second-order differential equations for the longitudinal field components E_z and H_z. Alternatively, if the radial components E_ρ and H_ρ are eliminated, we obtain a system of four first-order differential equations for the tangential field components E_z, E_ϕ, H_z, and H_ϕ.

First consider the coupled wave equation formulation. It is convenient to define a normalized magnetic field $h = Z_0 H$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Solving (3a), (3b), (3d), and (3e) for E_ρ, E_ϕ, h_ρ, and h_ϕ gives

$$E_\rho = \frac{1}{k_{t1}^2}\left[\frac{mk_0}{\rho} h_z - j\beta \frac{dE_z}{d\rho}\right] \tag{4a}$$

$$h_\phi = \frac{1}{k_{t1}^2}\left[\frac{m\beta}{\rho} h_z - jk_0\epsilon_1 \frac{dE_z}{d\rho}\right] \tag{4b}$$

$$E_\phi = \frac{1}{k_{t2}^2}\left[\frac{m\beta}{\rho} E_z + jk_0 \frac{dh_z}{d\rho}\right] \tag{4c}$$

$$h_\rho = \frac{1}{k_{t2}^2}\left[\frac{-mk_0\epsilon_2}{\rho} E_z - j\beta \frac{dh_z}{d\rho}\right] \tag{4d}$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ is the free-space wavenumber and $k_{tn}^2 = k_0^2\epsilon_n(\rho) - \beta^2$, $n = 1,2$, is the transverse wavenumber. Substituting the expressions for E_ρ, E_ϕ, h_ρ, and h_ϕ given by equations (4) into (3c) and (3f) and making a change of variable from ρ to a normalized radius $r = \rho/a$ results in the following pair of coupled differential equations for E_z and h_z:

$$E_z'' + f_1(r)E_z' + \Lambda^2 g_1(r)E_z = p_2(r)h_z' + q_2(r)h_z \tag{5a}$$

$$h_z'' + f_2(r)h_z' + \Lambda^2 g_2(r)h_z = p_1(r)E_z' + q_1(r)E_z, \tag{5b}$$

where $' = d/dr$, $\Lambda^2 = (k_0 a)^2$, $\kappa = \beta/k_0$, and

$$f_1(r) = \frac{1}{r} - \frac{\kappa^2 \epsilon_1'(r)}{\epsilon_1(r)\left[\epsilon_1(r) - \kappa^2\right]} \tag{6a}$$

$$f_2(r) = \frac{1}{r} - \frac{\epsilon_2'(r)}{\epsilon_2(r) - \kappa^2} \tag{6b}$$

$$g_1(r) = \frac{\epsilon_3(r)}{\epsilon_1(r)}\left[\epsilon_1(r) - \kappa^2\right]\left[1 - \frac{m^2 \epsilon_2(r)}{\Lambda^2 \epsilon_3(r)\left[\epsilon_2(r) - \kappa^2\right] r^2}\right] \tag{6c}$$

$$g_2(r) = \left[\epsilon_2(r) - \kappa^2\right]\left[1 - \frac{m^2}{\Lambda^2\left[\epsilon_1(r) - \kappa^2\right] r^2}\right] \tag{6d}$$

$$p_1(r) = \frac{jm\kappa}{r}\left[\frac{\epsilon_1(r) - \epsilon_2(r)}{\epsilon_1(r) - \kappa^2}\right] \tag{6e}$$

$$p_2(r) = -\frac{jm\kappa}{\epsilon_1(r) r}\left[\frac{\epsilon_2(r) - \epsilon_1(r)}{\epsilon_2(r) - \kappa^2}\right] \tag{6f}$$

$$q_1(r) = -\frac{jm\kappa}{r}\left[\frac{\epsilon_2'(r)}{\epsilon_2(r) - \kappa^2}\right] \tag{6g}$$

$$q_2(r) = \frac{jm\kappa}{\epsilon_1(r) r}\left[\frac{\epsilon_1'(r)}{\epsilon_1(r) - \kappa^2}\right]. \tag{6h}$$

The equations for E_z and h_z become uncoupled for three particular cases. For the so-called meridional modes m is equal to zero and therefore, from (6e)–(6h), so are the functions $p_1(r)$, $p_2(r)$, $q_1(r)$, and $q_2(r)$. For isotropic and uniaxial step-index fibers, ϵ_1 and ϵ_2 are equal and constant and again from (6e)–(6h) the functions $p_1(r)$, $p_2(r)$, $q_1(r)$, and $q_2(r)$ are zero.

In general a solution of equations (5) for arbitrary permittivity profiles is not possible. It is possible to obtain a fourth-order differential equation for either E_z or h_z by eliminating h_z or E_z from equations (5). However, the complexity of the resulting equation precludes the determination of a solution. For meridional modes a direct series solution of the uncoupled equations is possible. However, because of the poles in the functions $f_1(r)$ and $f_2(r)$ the resulting series solution will not be convergent for the entire core region. An exact solution of equations (5) is possible only for the case of a step-index fiber. For either an isotropic or a uniaxial step-index fiber the coupled equations simplify to Bessel's differential equation.

In order to find an analytic solution of equations (5) some assumptions must be made. First, the cladding is neglected and the core is assumed to extend to infinity. This eliminates the need to impose boundary conditions on the solution at the core–cladding boundary. Second, the permittivities are assumed to be slowly varying functions of r over a distance of several wavelengths. This is equivalent to assuming $\epsilon_i'(r) \approx 0$. For the case of either an isotropic or a uniaxial graded-index fiber, $\epsilon_1(r) = \epsilon_2(r)$, application of the second assumption to equations (5) results in the following equations for E_z and h_z:

$$E_z'' + \frac{1}{r}E_z' + \Lambda^2 g_1(r) E_z = 0 \tag{7a}$$

$$h_z'' + \frac{1}{r}h_z' + \Lambda^2 g_2(r) h_z = 0 \tag{7b}$$

where $g_1(r)$ and $g_2(r)$ are given by

$$g_1(r) = \frac{\epsilon_3(r)}{\epsilon_1(r)}\left[\epsilon_1(r) - \kappa^2\right] - \frac{m^2}{\Lambda^2 r^2} \tag{8a}$$

$$g_2(r) = \epsilon_1(r) - \kappa^2 - \frac{m^2}{\Lambda^2 r^2}. \tag{8b}$$

For the case of a biaxial graded-index fiber, $\epsilon_1(r) \neq \epsilon_2(r)$, the previous assumption does not cause equations (5) to uncouple since $p_1(r)$ and $p_2(r)$ are not identically equal to zero.

Equations (7) can be solved easily using the well-known WKB solution method [6], [7], [16]. The solutions obtained using this method are not solutions of the full vector problem given by equations (5) but rather are solutions to a related scalar problem given by equations (7). However, the vector solutions can be obtained by renumbering the solutions to the scalar problem [10].

For the case of a biaxial graded-index fiber the WKB solution method can be applied blindly to equations (5) to determine the first term in the WKB expansion (higher order terms are coupled). However, for biaxial graded-index fibers, the term representing the phase of the WKB solution contains a pole in addition to the one at $r = 0$ which, for most permittivity profiles, lies in the core region. Therefore, it is not reasonable to assume that the WKB phase condition remains valid in this situation. In order to solve Maxwell's equations for the case of a biaxial graded-index fiber an alternative formulation should be used.

Instead of eliminating the transverse field components from equations (3), eliminate the radial field components E_ρ and H_ρ and write the remaining four equations as a system of four first-order differential equations in terms of the tangential components [15]. From the two algebraic equations, (3a) and (3d), the radial components can be written in terms of the tangential components as

$$E_\rho = \frac{1}{\omega \epsilon_0 \epsilon_1}\left[\frac{m}{\rho}H_z + \beta H_\phi\right] \tag{9a}$$

$$H_\rho = -\frac{1}{\omega \mu_0}\left[\frac{m}{\rho}E_z + \beta E_\phi\right]. \tag{9b}$$

Using equations (9), the four remaining equations can be written as

$$\frac{dE_z}{ds} = -j\frac{m\kappa}{s\epsilon_1}h_z + \frac{j}{s\epsilon_1}(\epsilon_1 - \kappa^2)(sh_\phi) \quad (10a)$$

$$\frac{d}{ds}(sE_\phi) = \frac{j}{s\epsilon_1}(m^2 - \epsilon_1 s^2)h_z + j\frac{m\kappa}{s\epsilon_1}(sh_\phi) \quad (10b)$$

$$\frac{dh_z}{ds} = j\frac{m\kappa}{s}E_z - \frac{j}{s}(\epsilon_2 - \kappa^2)(sE_\phi) \quad (10c)$$

$$\frac{d}{ds}(sh_\phi) = -\frac{j}{s}(m^2 - \epsilon_3 s^2)E_z - j\frac{m\kappa}{s}(sE_\phi) \quad (10d)$$

where a change of variable from ρ to a normalized radius $s = k_0\rho$ has been made. Equations (10) can be written in matrix form as

$$\frac{d\boldsymbol{u}}{ds} = \frac{1}{s}\boldsymbol{A}(s)\boldsymbol{u} \quad (11a)$$

where

$$\boldsymbol{u} = (E_z \quad sE_\phi \quad h_z \quad sh_\phi)^T \quad (11b)$$

and

$$\boldsymbol{A}(s) = \begin{pmatrix} 0 & 0 & -j\frac{m\kappa}{\epsilon_1} & \frac{j}{\epsilon_1}(\epsilon_1 - \kappa^2) \\ 0 & 0 & \frac{j}{\epsilon_1}(m^2 - \epsilon_1 s^2) & j\frac{m\kappa}{\epsilon_1} \\ jm\kappa & -j(\epsilon_2 - \kappa^2) & 0 & 0 \\ -j(m^2 - \epsilon_3 s^2) & -jm\kappa & 0 & 0 \end{pmatrix}. \quad (11c)$$

For the special case of meridional modes, $m = 0$, equations (10) can be separated into two systems, each containing two equations. The first set, corresponding to transverse magnetic modes (TM), can be written in matrix form as

$$\frac{d\boldsymbol{u}^{(\mathrm{TM})}}{ds} = \frac{1}{s}\boldsymbol{A}^{(\mathrm{TM})}(s)\boldsymbol{u}^{(\mathrm{TM})} \quad (12a)$$

where

$$\boldsymbol{u}^{(\mathrm{TM})} = (E_z \quad sh_\phi)^T \quad (12b)$$

and

$$\boldsymbol{A}^{(\mathrm{TM})}(s) = \begin{pmatrix} 0 & \frac{j}{\epsilon_1}(\epsilon_1 - \kappa^2) \\ j\epsilon_3 s^2 & 0 \end{pmatrix}. \quad (12c)$$

The second set, corresponding to transverse electric modes (TE), can be written as

$$\frac{d\boldsymbol{u}^{(\mathrm{TE})}}{ds} = \frac{1}{s}\boldsymbol{A}^{(\mathrm{TE})}(s)\boldsymbol{u}^{(\mathrm{TE})} \quad (13a)$$

where

$$\boldsymbol{u}^{(\mathrm{TE})} = (h_z \quad sE_\phi)^T \quad (13b)$$

and

$$\boldsymbol{A}^{(\mathrm{TE})}(s) = \begin{pmatrix} 0 & -j(\epsilon_2 - \kappa^2) \\ -js^2 & 0 \end{pmatrix}. \quad (13c)$$

The only known exact solutions of the matrix equation are for the cases of an isotropic and a uniaxial step-index fiber [14], [15]. These solutions are identical to the exact solutions of the wave equation formulation.

It is not readily apparent that the matrix equation is easier to solve than the wave equation formulation. As was mentioned earlier, a series solution for the wave equation formulation is possible only when the equations are uncoupled. However, for the meridonal modes of a graded-index fiber no series solution will be convergent for the entire core region. In contrast, the system matrix $A(s)$ does not have any poles in the core region and therefore the series solutions will be convergent in the entire core region.

While the form of $A(s)$ guarantees a convergent series solution the series may not converge rapidly enough to use it in numerical computations. An alternative solution method is asymptotic partitioning of systems of equations [17]. This method involves the transformation of a system of linear first-order differential equations into a system of equations whose solutions are easier to find. The form of the solution method presented in the next section is based on the expansion of the general system matrix $A(x)$ in terms of positive powers of x, in contrast to the usual form, where the expansion is in terms of powers of $1/x$ (see e.g. [17])

III. Matrix Partitioning

Consider the following system of N linear differential equations:

$$\frac{d\boldsymbol{u}}{dx} = \frac{1}{x^q}\boldsymbol{A}(x)\boldsymbol{u}(x) \qquad \text{as } x \to 0 \quad (14)$$

where $\boldsymbol{u}$ is a column vector, q is an integer greater than or equal to 1, and $A(x)$ is an $N \times N$ matrix given by

$$A(x) = \sum_{n=0}^{\infty} A_n x^n \qquad \text{as } x \to 0. \quad (15)$$

It is possible to simplify this system of equations by transforming them into some special differential equations whose solutions are easier to find. Let

$$\boldsymbol{u}(x) = \boldsymbol{P}(x)\boldsymbol{v}(x) \quad (16)$$

where $\boldsymbol{v}$ is a column vector and $\boldsymbol{P}(x)$ is an $N \times N$ nonsingular matrix. Using (16), the original problem given by (14) can be transformed into

$$\frac{d\boldsymbol{v}}{dx} = \frac{1}{x^q}\boldsymbol{B}(x)\boldsymbol{v}(x) \quad (17)$$

where

$$B(x)=P(x)^{-1}\left[A(x)P(x)-x^{q}\frac{dP(x)}{dx}\right] \quad (18)$$

or, more conveniently,

$$x^{q}\frac{dP(x)}{dx}=A(x)P(x)-P(x)B(x). \quad (19)$$

The matrix $P(x)$ is chosen so that $B(x)$ has a convenient form, either the diagonal or the Jordan canonical form. If $B(x)$ has either of these forms, the solution of the transformed system for v is trivially obtained. For example, if $A(x)$ is a constant matrix, then $P(x)$ is also a constant matrix and (18) is simply a similarity transformation. This implies $P(x)$ is chosen so that $B(x)$ is either the diagonal or the Jordan canonical form of $A(x)$. In general, when $A(x)$ is not a constant matrix, $P(x)$ is not a constant matrix and it is not clear from either (18) or (19) how $P(x)$ should be chosen so that $B(x)$ has the desired form.

In order to develop a procedure to find $B(x)$ and $P(x)$, start by expanding them as the following Taylor series:

$$B(x)=\sum_{n=0}^{\infty}B_n x^n \qquad \text{as } x\to 0$$

$$P(x)=\sum_{n=0}^{\infty}P_n x^n \qquad \text{as } x\to 0 \quad (20)$$

where in general B_0 is a Jordan canonical matrix and B_n is a diagonal matrix. Substituting equations (20) into (19) and equating like powers of x gives

$$A_0P_0-P_0B_0=0 \quad (21)$$

for x^0 and

$$(n-q+1)P_{n-q+1}=\sum_{l=0}^{n}(A_lP_{n-l}-P_lB_{n-l}) \quad (22)$$

for x^n, $n\geqslant 1$, where $P_{n-q+1}=0$ for $n-q+1<0$. Equations (21) and (22) define an iterative procedure to find the coefficient matrices for the series expansions of $B(x)$ and $P(x)$ so that either (18) or (19) is satisfied. Equation (21) can be rewritten as

$$B_0=P_0^{-1}A_0P_0 \quad (23)$$

which implies P_0 is chosen so that B_0 is either the diagonal or the Jordan canonical form of A_0. With some algebraic manipulations, (22) can be written more conveniently as

$$B_0W_n-W_nB_0=(n-q+1)W_{n-q+1}+B_n-F_n \quad (24)$$

where the matrices W_n and F_n are defined as

$$W_n=P_0^{-1}P_n \quad (25)$$

and

$$F_n=P_0^{-1}A_nP_0+P_0^{-1}\sum_{l=1}^{n-1}(A_{n-l}P_l-P_lB_{n-l}). \quad (26)$$

Notice that the unknowns in (24) are the matrices B_n and W_n and that the matrices W_{n-q+1} and F_n depend solely on matrices found in previous iterations. Since by definition B_n is a diagonal matrix, (24) can be solved easily for B_n and W_n by setting the diagonal elements of B_n and F_n equal to each other and then solving for W_n from what remains of (24). Since the form of B_0 is known in advance, an explicit solution in terms of B_0 and F_n can be found for W_n.

Consider the special case where B_0 is a diagonal matrix and $q=1$. This corresponds to the form of the matrix differential equation (11) which we want to solve. While it is not obvious from (11c) that, for this particular problem, B_0 will be a diagonal matrix, it will be shown later that it is possible to choose P_0 such that this is true.

When B_0 is a diagonal matrix the expression $B_0W_n-W_nB_0$ has zeros along its main diagonal and does not depend upon the elements along the main diagonal of W_n. The solution to (24) can be easily written as

$$(B_n)_{ij}=\begin{cases}(F_n)_{ii}, & i=j\\ 0, & i\neq j\end{cases} \quad (27)$$

and

$$(W_n)_{ij}=\begin{cases}0, & i=j\\ -\dfrac{1}{\lambda_i-\lambda_j-n}(F_n)_{ij}, & i\neq j\end{cases} \quad (28)$$

where λ_i, $i=1,2,\cdots,N$, are the eigenvalues of A_0. One potential problem exists with this solution. If $\lambda_i-\lambda_j-n=0$ and $(F_n)_{ij}\neq 0$ for some particular values of i, j, and n, then it may not be possible to find W_n and a solution may therefore not be possible.

Consider a biaxial graded-index fiber with permittivity profiles of the power law type given by

$$\epsilon_i(r)=\epsilon_i(1-2\Delta_i r^{\alpha_i}), \qquad i=1,2,3 \quad (29)$$

where $\epsilon_i=\epsilon_i(0)$ and $\Delta_i=(\epsilon_i-\epsilon_c)/2\epsilon_i$. Since the choice of the coordinate system requires $\epsilon_1(0)=\epsilon_2(0)$, the definition of Δ_i requires that Δ_1 and Δ_2 be equal. Then, for this choice of permittivity profiles, $\epsilon_1(r)$ and $\epsilon_2(r)$ are not equal only when $\alpha_1\neq\alpha_2$. The case of a step-index fiber exists as a special case to the power law profiles in the limit as $\alpha_i\to\infty$, or, equivalently, by setting $\Delta_i=0$.

Now let us solve the matrix equation for the transverse modes in a biaxial fiber where the permittivity profiles are parabolic. The relative permittivity profiles can be written in terms of the normalized radius $s=k_0\rho=(k_0a)r$ as

$$\epsilon_i(s)=\epsilon_i(1-2\Delta_i^0 s^2), \qquad i=1,2,3 \quad (30)$$

where $\Delta_i^0=\Delta_i/(k_0a)^2$. Strictly speaking, this choice for $\epsilon_i(s)$ does not produce a biaxial fiber since by definition $\epsilon_1=\epsilon_2$ and $\Delta_1^0=\Delta_2^0$. Since $\epsilon_1(s)$ and $\epsilon_2(s)$ do not appear together in the matrix equations for the transverse modes, (12) and (13), it is not necessary to set ϵ_2 equal to ϵ_1. However, in the final result it is necessary to replace ϵ_2 by ϵ_1 and either set $\Delta_1^0=0$ and obtain the solution for a biaxial fiber where $\epsilon_1(s)$ is constant and $\epsilon_2(s)$ is parabolic or set $\Delta_2^0=0$ and obtain the solution for the case where $\epsilon_1(s)$ is parabolic and $\epsilon_2(s)$ is constant. The first term in the series expansions for $A^{(\mathrm{TM})}(s)$ and $A^{(\mathrm{TE})}(s)$ is given by

$$A_0^i=\begin{pmatrix}0 & \tau\\ 0 & 0\end{pmatrix} \quad (31a)$$

where

$$\tau = \begin{cases} \dfrac{j}{\epsilon_1}(\epsilon_1 - \kappa^2), & i = \text{TM} \\ -j(\epsilon_2 - \kappa^2), & i = \text{TE} \end{cases}. \tag{31b}$$

Since the two eigenvalues of A_0 are both equal to zero it is not possible to find a nonzero P_0 such that B_0 is a diagonal matrix. Instead, P_0 must be chosen so that B_0 is a Jordan canonical matrix. For A_0 as given by (31), choose P_0 as

$$P_0 = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix} \tag{32}$$

so that

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{33}$$

is a Jordan canonical matrix. After four iterations, the solution for the TM case is found to be

$$E_z = \frac{j}{\epsilon_3} k_{N1}^2 C_1 \left\{ 1 - \frac{\epsilon_3}{\epsilon_1} \frac{k_{N1}^2}{4} s^2 + \left[\left(\frac{\epsilon_3}{\epsilon_1} \right)^2 \frac{k_{N1}^4}{64} + \frac{\epsilon_3}{\epsilon_1} \Delta_3^0 \frac{k_{N1}^2}{8} \right] s^4 \right\} e^{(\epsilon_3/\epsilon_1)\Delta_1^0 \kappa^2 s^4/4} \tag{34a}$$

$$sh_\phi = -C_1 \left\{ \frac{\epsilon_3}{\epsilon_1} \frac{k_{N1}^2}{2} s^2 - \left[\left(\frac{\epsilon_3}{\epsilon_1} \right)^2 \frac{k_{N1}^4}{16} + \frac{\epsilon_3}{\epsilon_1} \Delta_3^0 \frac{k_{N1}^2}{2} \right] s^4 \right\} e^{(\epsilon_3/\epsilon_1)\Delta_1^0 \kappa^2 s^4/4} \tag{34b}$$

where $k_{N1}^2 = \epsilon_1 - \kappa^2$ and C_1 is a constant. The solution for the TE case is

$$h_z = -jk_{N2}^2 C_1 \left[1 - \frac{k_{N2}^2}{4} s^2 + \frac{k_{N2}^4}{64} s^2 \right] e^{\epsilon_2 \Delta_2^0 s^4/4} \tag{35a}$$

$$sE_\phi = -C_1 \left[\frac{k_{N2}^2}{2} s^2 - \frac{k_{N2}^4}{16} s^4 \right] e^{\epsilon_2 \Delta_2^0 s^4/4} \tag{35b}$$

where $k_{N2}^2 = \epsilon_2 - \kappa^2$.

Now consider the solution of the matrix equation for hybrid modes. For all permittivity profiles the first term in the series expansion of $A(s)$ is

$$A_0 = \begin{pmatrix} 0 & 0 & -j\dfrac{m\kappa}{\epsilon_1} & j\dfrac{k_{N1}^2}{\epsilon_1} \\ 0 & 0 & \dfrac{jm^2}{\epsilon_1} & j\dfrac{m\kappa}{\epsilon_1} \\ jm\kappa & -jk_{N1}^2 & 0 & 0 \\ -jm^2 & -jm\kappa & 0 & 0 \end{pmatrix} \tag{36}$$

where $k_{N1}^2 = \epsilon_1 - \kappa^2$ and the eigenvalues of A_0 are $\pm m$, $m \neq 0$. Since the eigenvalues are repeated, in general, the choice for P_0 should at best cause B_0 to be a Jordan canonical matrix. This is the only restriction placed on the form of P_0 by the solution method. Any P_0 which causes B_0 to be a Jordan canonical matrix can be expected to result in a valid solution. Since it is possible for several different choices of P_0 to satisfy this condition, conceivably there may exist several possible mathematical solutions to the problem.

Since the solution for a step-index fiber exists as a special case of the solution for a graded-index fiber, it is reasonable to choose P_0 based on the knowledge of the exact solution for a step-index fiber. From the wave equation formulation we know that for a step-index fiber the differential equations for E_z and h_z become uncoupled and the resulting equations can be solved independently of each other. This suggests that for the case of a step-index fiber $P(s)$ and hence P_0 should have a form such that two of the four elements in the solution of the vector $v(s)$ should contribute to E_z but not h_z while the remaining two elements contribute to h_z only. If P_0 is chosen as

$$P_0 = \begin{pmatrix} k_{N1}^2 & 0 & k_{N1}^2 & 0 \\ m\kappa & jm & m\kappa & -jm \\ 0 & k_{N1}^2 & 0 & k_{N1}^2 \\ -jm\epsilon_1 & m\kappa & jm\epsilon_1 & m\kappa \end{pmatrix} \tag{37}$$

then for a step-index fiber E_z and h_z are at least uncoupled for the lowest order solution where $P(s) = P_0$.

Using P_0 given by (37), B_0 is given by

$$B_0 = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix}. \tag{38}$$

Since B_0 is a diagonal matrix instead of a Jordan canonical matrix, as was the case for the transverse modes, (28) can be used to find W_n. Recall that this solution for W_n may cause some elements of W_n to be undefined. In particular, for this problem the elements in the third and fourth columns of both W_{2k} and P_{2k} are undefined when $m = 1, 2, \cdots, k$. However, owing to the structure of the various matrices and the order of multiplication in the definitions of W_n and F_n, these undefined elements remain in the third and fourth columns of all resulting matrices. In the final solution these undefined elements can be dropped since they contribute only to the two solutions which are not finite at $s = 0$.

With P_0 and B_0 given by (37) and (38), respectively, the general form of the solution to (11a) which is finite at $s = 0$ is given by

$$\begin{pmatrix} E_z \\ sE_\phi \\ h_z \\ sh_\phi \end{pmatrix} = s^m \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \\ P_{31}(s) & P_{32}(s) \\ P_{41}(s) & P_{42}(s) \end{pmatrix} \begin{pmatrix} C_1 e^{\lambda_1(s)} \\ C_2 e^{\lambda_2(s)} \end{pmatrix} \tag{39a}$$

where

$$P_{ij}(s) = \sum_{n=0}^{N} (P_n)_{ij} s^n, \qquad i = 1,2,3,4; \quad j = 1,2 \tag{39b}$$

$$\lambda_i(s) = \sum_{n=1}^{N} (B_n)_{ii} \frac{s^n}{n}, \qquad i = 1,2 \tag{39c}$$

and N is the number of iterations.

For a biaxial graded-index fiber where $\epsilon_1(s)$ and $\epsilon_3(s)$ have a parabolic profile given by (30) and $\epsilon_2(s)$ is a constant, after two iterations the following expressions are found for $\lambda_i(s)$ and $P_{ij}(s)$:

$$\lambda_1(s) = -\frac{1}{4m}\left[\frac{\epsilon_3}{\epsilon_1}(\epsilon_1-\kappa^2)+\frac{m^2\kappa^2}{\epsilon_1-\kappa^2}(2\Delta_1^0)\right]s^2 \tag{40a}$$

$$\lambda_2(s) = -\frac{1}{4m}\left[(\epsilon_1-\kappa^2)-\frac{m^2\epsilon_1}{\epsilon_1-\kappa^2}(2\Delta_1^0)\right]s^2 \tag{40b}$$

$$P_{11}(s) = (\epsilon_1-\kappa^2) + \frac{1}{4m(m+1)}\left[\frac{\epsilon_3}{\epsilon_1}(\epsilon_1-\kappa^2)^2 - m^2\kappa^2(2\Delta_1^0)\right]s^2 \tag{40c}$$

$$P_{12}(s) = -j\frac{m\kappa}{4}\left(\frac{m+2}{m+1}\right)(2\Delta_1^0)s^2 \tag{40d}$$

$$P_{21}(s) = m\kappa + \frac{\kappa}{4(m+1)}\left\{\frac{\epsilon_3}{\epsilon_1}(\epsilon_1-\kappa^2) + \frac{m^2(2\Delta_1^0)}{\epsilon_1-\kappa^2}\left[(\epsilon_1-\kappa^2)+(m+1)\epsilon_1\right]\right\}s^2 \tag{40e}$$

$$P_{22}(s) = jm - \frac{j}{4(m+1)}\left\{(\epsilon_1-\kappa^2) + \frac{m^2(2\Delta_1^0)}{\epsilon_1-\kappa^2}\left[(m+1)\kappa^2-(\epsilon_1-\kappa^2)\right]\right\}s^2 \tag{40f}$$

$$P_{31}(s) = -j\frac{m^2\epsilon_1\kappa}{4(m+1)}(2\Delta_1^0)s^2 \tag{40g}$$

$$P_{32}(s) = (\epsilon_1-\kappa^2) + \frac{1}{4m(m+1)}\left[(\epsilon_1-\kappa^2)^2 - m^2\epsilon_1(2\Delta_1^0)\right]s^2 \tag{40h}$$

$$P_{41}(s) = -jm\epsilon_1 + \frac{j\epsilon_1}{4(m+1)} \cdot\left[\frac{\epsilon_3}{\epsilon_1}(\epsilon_1-\kappa^2) - \frac{m^2(m+1)\kappa^2}{\epsilon_1-\kappa^2}(2\Delta_1^0)\right]s^2 \tag{40i}$$

$$P_{42}(s) = m\kappa + \frac{\kappa}{4(m+1)} \cdot\left[(\epsilon_1-\kappa^2) - \frac{m^2(m+1)\epsilon_1}{\epsilon_1-\kappa^2}(2\Delta_1^0)\right]s^2. \tag{40j}$$

This should not be considered an accurate solution for $\boldsymbol{u}(s)$ since the term Δ_3^0 does not appear anywhere in equations (40). This solution is identical to the solution obtained after two iterations for a biaxial graded-index fiber where $\epsilon_1(s)$ has a parabolic profile and $\epsilon_2(s)$ and $\epsilon_3(s)$ are constant. Since Δ_3^0 appears only in the matrix A_4, at least four iterations must be performed in order to obtain the effects of a nonconstant $\epsilon_3(s)$.

The solution for a uniaxial or a step index-fiber can be obtained from equations (40) by setting Δ_1^0 (and Δ_3^0) equal to zero. Notice that setting Δ_1^0 equal to zero causes $P_{12}(s)$ and $P_{31}(s)$ to be set equal to zero. This corresponds to the decoupling of the differential equations which occurs in the wave equation formulation for the case of a step-index fiber.

From numerical results, it appears that the functions $\lambda_i(s)$ and $P_{ij}(s)$ given in equations (40) are monotonic. This indicates that the solutions for the various field components will not have an oscillatory behavior. Consequently, for a given value of m, only the mode with the lowest cutoff frequency will be found.

IV. Numerical Results

The propagation constants are found by requiring the tangential components of the electric and magnetic fields in the core and cladding to be continuous at the core–cladding interface (i.e., at $\rho = a$ or $s = k_0a$). In the cladding, the fields can be found easily from the wave equation formulation. Since the permittivity of the cladding, $\epsilon_0\epsilon_c$, is a constant, (5a) and (5b) reduce to Bessel's equation. In order for the fields in the cladding to decay exponentially as ρ approaches infinity, choose K_m, the modified Bessel function of the second kind, as the solution. Using (4b) and (4c), the tangential field components in the cladding can be written as

$$\begin{pmatrix} E_z \\ sE_\phi \\ h_z \\ sh_\phi \end{pmatrix} = \begin{pmatrix} K_m(\gamma Ns) & 0 \\ -\frac{m\kappa}{\gamma_N^2}K_m(\gamma Ns) & -j\frac{s}{\gamma N}K_m'(\gamma Ns) \\ 0 & K_m(\gamma Ns) \\ j\frac{\epsilon_c s}{\gamma N}K_m'(\gamma Ns) & -\frac{m\kappa}{\gamma_N^2}K_m(\gamma Ns) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \tag{41}$$

where A and B are constants and $\gamma_N^2 = \kappa^2 - \epsilon_c$. Using (39a) and (41), the boundary conditions at $s = k_0a$ ($\rho = a$) are satisfied, provided that

$$\begin{pmatrix} P_{11} & P_{12} & -K_m & 0 \\ P_{21} & P_{22} & \frac{m\kappa}{\gamma_N^2}K_m & j\frac{k_0a}{\gamma N}K_m' \\ P_{31} & P_{32} & 0 & -K_m \\ P_{41} & P_{42} & -j\frac{\epsilon_c k_0 a}{\gamma N}K_m' & \frac{m\kappa}{\gamma_N^2}K_m \end{pmatrix} \cdot \begin{pmatrix} (k_0a)^m C_1 e^{\lambda_1} \\ (k_0a)^m C_2 e^{\lambda_2} \\ A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{42}$$

where $P_{ij} = P_{ij}(k_0a)$, $\lambda_i = \lambda_i(k_0a)$, $K_m = K_m(k_0a\gamma_N)$, and $K_m' = dKm(k_0a\gamma_N)/d(k_0a\gamma_N)$. The normalized propagation constant, κ, as a functions of the normalized wavenumber, k_0a, is determined by finding those values of κ and k_0a such that the determinant of the matrix in (42) is identically equal to zero.

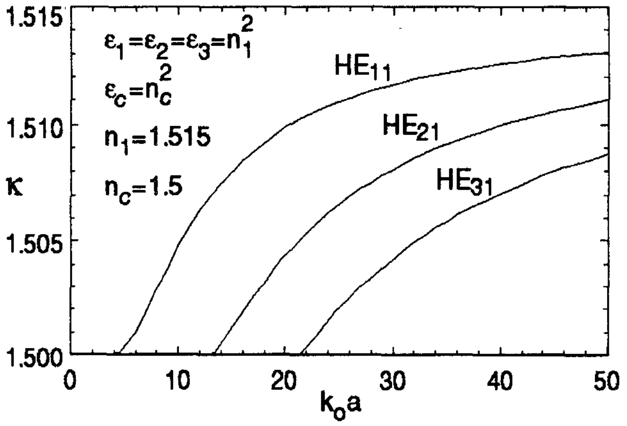

Fig. 2. Normalized propagation constants for the HE_{11}, HE_{21}, and HE_{31} modes in a biaxial graded-index fiber where $\epsilon_1(s)$ and $\epsilon_3(s)$ have the parabolic profile given by eq. (30) and $\epsilon_2(s)$ is a constant (i.e., $\Delta_2^0 = 0$).

As was previously stated, the solution for the biaxial graded-index fiber given by equations (40) does not include the effects of a nonconstant $\epsilon_3(s)$. Obtaining a more accurate solution requires performing more than four iterations. Instead of deriving algebraic equations for the elements of $\boldsymbol{F}_n$, $\boldsymbol{B}_n$, $\boldsymbol{W}_n$, and $\boldsymbol{P}_n$, the values of these matrices can be determined numerically if the values of m, κ, and k_0a are known in advance. One potential difficulty with this method comes from the undefined elements in $\boldsymbol{W}_n$ and $\boldsymbol{P}_n$. Since these elements contribute only to the solutions which are unbounded at $s=0$ they can be set equal to zero without affecting the final solution. The ability to do this appears to depend upon the form of $A(s)$ and the ordering of the eigenvalues of $\boldsymbol{A}_0$ in $\boldsymbol{B}_0$.

Asymptotic partitioning was used to solve the matrix equation for several types of fibers. For the case of a step-index fiber a comparison was made between the propagation constants determined using asymptotic partitioning and those determined using the exact solution. For transverse modes the asymptotic solutions were in poor agreement with the exact solutions. Since for transverse modes in a step-index fiber asymptotic partitioning produces a series solution, the poor agreement can be attributed to using too few terms in the series expansion of the exact solution. For hybrid modes there was a much better agreement between the asymptotic solutions and the exact solutions. In particular, for a step-index fiber the asymptotic and the exact solutions produced almost identical values for the propagation constants of the HE_{11} mode.

Fig. 2 is a plot of the normalized propagation constant for the HE_{11}, HE_{21}, and HE_{31} modes in a biaxial graded-index fiber where $\epsilon_1(s)$ and $\epsilon_3(s)$ have a parabolic profile and $\epsilon_2(s)$ is a constant.

V. Conclusions

For both the wave equation formulation and the matrix equation, exact solutions are known only for the cases of an isotropic step-index fiber and a uniaxial step-index fiber. For isotropic and uniaxial graded-index fibers the wave equation formulation can be solved approximately using WKB analysis. For a biaxial graded-index fiber WKB analysis cannot be used on the wave equation formulation. Asymptotic partitioning can be used to solve the matrix equation for all types of permittivity profiles. For meridional modes, asymptotic partitioning appears to generate the series solution for the matrix differential equation. For hybrid modes, the solutions produced by asymptotic partitioning have a form such that for a given value of m only the mode with the lowest cutoff frequency can be found. A nice feature of the asymptotic solutions is that they remain valid all the way down to cutoff.

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